ON THE SOLUTION OF CERTAIN PROBLEMS IN MAGNETOHYDRODYNAMICS WITH ANISOTROPIC CONDUCTIVITY

(O RESHENII NEKOTORYKH ZADACH MAGNITNOI GIDRODINAMIKI PRI ANIZOTROPNOI PROVODIMOSTI)

PMM Vol.26, No.3, 1962, pp. 530-541

G.A. LIUBIMOV
(Moscow)

(Received January 25, 1962)

From the generalized Ohm's law, with certain assumptions concerning the nature of the problems under consideration, we obtain a vector equation describing the change in magnetic field, independent of the equations of mechanics. If the parameters of the problem are assumed to be independent of one of the coordinates, then this equation reduces to a system of two scalar equations. The properties of certain particular solutions of this system are described. For the case of small currents and large external magnetic fields, when to the first approximation the induced fields may be neglected, the solution of the resulting system can be found in the form of a series in the small parameter $\lambda = 4\pi I/cH_0$ (I is the total current flowing in the system, H_0 is the external field). As an example we consider the problem of the effect of anisotropic conductivity on the flow of gas in a channel with crossed electric and magnetic fields.

1. We shall consider the flow of a rarefied ionized gas in a strong external magnetic field, and we shall assume that the parameter

$$\omega\tau \gtrsim 1$$

(ω is the Larmor frequency of electrons, τ is the time between collisions of electrons). Under these conditions the equation expressing the generalized Ohm's law has the form

$$\mathbf{j} = \sigma\left(\mathbf{E} + \frac{1}{c}\mathbf{v}\times\mathbf{H}\right) - \frac{\omega\tau}{H}\mathbf{j}\times\mathbf{H} + \frac{\sigma}{ne}\operatorname{grad} p \qquad (1.1)$$

Media in which $\omega\tau \gtrsim 1$ are known as media with anisotropic conductivity.

To specify the motion of the gas and the laws of current flow we have to supplement Equation (1.1) with the equations of mechanics and Maxwell's equations.

If the degree of ionization is slight, so that the number of neutral particles n_a considerably exceeds the number of charged particles n, i.e. $n_a \gg n$, then the term involving the gradient of electron pressure in Equation (1.1) may be neglected.

In a number of problems connected with the motion of a conducting gas in an electromagnetic field, the magnetic Reynolds' number R_m is small on account of the low conductivity of the medium and the small velocities of motion. If at the same time strong electric currents are created in the gas by external sources, so that the following relation holds:

$$\lambda = \frac{4\pi I}{cH} \gg R_m \qquad \left(R_m = \frac{4\pi\sigma VL}{c^2}\right) \tag{1.2}$$

(I is the total current in the system), then the induced currents may be neglected ($j \gg \sigma Hv/c$). Such conditions obtain, for example, in problems concerning the setting in motion of a conducting gas in a channel by external crossed electric and magnetic fields. In the initial stage of the motion, when the velocity of the gas is still far from the limiting velocity $u^* \sim cE/H$, the electric currents in the system are caused by the external potential difference, whilst the external electrical field is many times greater than the induction field ($E \gg vH/c$) and the relation (1.2) is valid when $R_m \ll 1$.

From now on it will be assumed that the inequalities (1.2) and $n_a \gg n$ are valid. Then the generalized Ohm's law takes the form

$$\mathbf{j} = \sigma\mathbf{E} - \frac{\omega\tau}{H}\mathbf{j} \times \mathbf{H} \tag{1.3}$$

It is obvious that under the specified assumptions Equation (1.3) and Maxwell's equations, determining the change in the electromagnetic field and the current field, do not depend on the equations of mechanics. Using this system to find the distribution of fields and currents, we can calculate the force acting on the gas and the energy supplied by the electromagnetic field, after which we can solve the mechanical problem of the motion of the gas in the specified field of force and with the specified supply of energy.

* We note that Equation (1.3) is exact if we consider the problem of determining the current field in a fixed conductor possessing anisotropic conductivity.

Applying the operator rot (curl) to Equation (1.3) and making use of

Maxwell's equations, we obtain the following equation, which describes the changes in the magnetic field:

$$\triangle \mathbf{H} - \alpha \left[(\mathbf{H}\nabla) \operatorname{rot} \mathbf{H} - (\operatorname{rot} \mathbf{H} \nabla) \mathbf{H}\right] = \frac{1}{\nu_m} \frac{\partial \mathbf{H}}{\partial t} \tag{1.4}$$

In these manipulations we have made use of the fact that

$$\sigma = \text{const}, \qquad \alpha \equiv \frac{\omega\tau}{H} = \text{const} \tag{1.5}$$

Equation (1.4) is analogous to the equation of induction in magnetohydrodynamics. With the boundary and initial conditions, respectively, formulated for $\mathbf{H}$, Equation (1.4) together with the condition div $\mathbf{H}$ = 0 determines the distribution of magnetic field. Knowing the solution of Equation (1.4), from Maxwell's equations and Ohm's law (1.3) we can find the distribution of all the electromagnetic quantities.

2. We shall consider problems which are stationary from the electrodynamic point of view. Then Equation (1.4) takes the form

$$\triangle \mathbf{H} - \alpha \left[(\mathbf{H}\nabla) \operatorname{rot} \mathbf{H} - (\operatorname{rot} \mathbf{H}\nabla) \mathbf{H}\right] = 0 \tag{2.1}$$

Having in view the application of our results to problems connected with the flow of gas in pipes and channels under the action of an external electromagnetic field, we shall seek solutions of Equation (2.1) which do not depend on the coordinate x. Moreover it will be assumed that the boundary conditions do not depend upon x. Under these assumptions it follows from Maxwell's equations that the projections of Equation (2.1) on the axes of y and z are equivalent to the relation E_x = const. The value of the constant E_x is connected (when all quantities are independent of x) with the conditions at infinity or at the outlet of the channel.

To obtain the system of equations which describes the distribution of magnetic field in this case, we can make use of the projection of Equation (1.3) on the axis of x, the projection of Equation (2.1) on the axis of x, and the equation div $\mathbf{H}$ = 0. This system will have the form

$$\frac{\partial H_z}{\partial y} - \frac{\partial H_y}{\partial z} + \alpha \left(\frac{\partial H_x}{\partial z} H_z + H_y \frac{\partial H_x}{\partial y}\right) = E_x = \text{const}$$

$$\triangle H_x - \alpha \left[H_y \triangle H_z - H_z \triangle H_y\right] = 0, \qquad \frac{\partial H_y}{\partial y} + \frac{\partial H_z}{\partial z} = 0 \tag{2.2}$$

For the sake of simplicity in what follows we shall assume that

$$E_x = 0 \tag{2.3}$$

i.e. that no accumulation of electric charge occurs at infinity along the axis of x. For circulatory flows of gas of the type of flow with homopolarity the condition (2.3) is automatically fulfilled.

Let us introduce the function $\Phi(yz)$ by the formulas

$$H_y = \frac{\partial \Phi}{\partial z}, \qquad H_z = -\frac{\partial \Phi}{\partial y} \tag{2.4}$$

The last of Equations (2.2) is then satisfied identically, whilst the first two reduce to the system

$$\begin{aligned} \Delta H_x + \alpha \left[\frac{\partial \Phi}{\partial z}\frac{\partial}{\partial y} - \frac{\partial \Phi}{\partial y}\frac{\partial}{\partial z} \right] \Delta \Phi = 0 \\ \Delta \Phi - \alpha \left[\frac{\partial \Phi}{\partial z}\frac{\partial}{\partial y} - \frac{\partial \Phi}{\partial y}\frac{\partial}{\partial z} \right] H_x = 0 \end{aligned} \tag{2.5}$$

If we consider the flow of gas along a channel, then the system (2.5) has to be solved with boundary conditions given in the yz plane on the contour of the channel. Moreover, on a part of the contour representing an insulator, the normal component of the current density must vanish, i.e.

$$j_n = \frac{c}{4\pi}\frac{\partial H_x}{\partial s} = 0, \quad \text{or} \quad H_x = \text{const} \quad \text{on insulators} \tag{2.6}$$

On a part of the contour representing an ideal conductor (electrode), on the other hand, the tangential component of the electric field must vanish, i.e.

$$\sigma E_s = j_s + \alpha (H_x j_n - H_n j_x) = 0, \quad \text{or} \quad \frac{c}{4\pi}\frac{\partial H_x}{\partial n} = \alpha (H_n j_x - H_x j_n) \left(\text{on conductors} \right) \tag{2.7}$$

Here s is the coordinate along the contour of the channel, $\mathbf{n}$ is the exterior normal to the contour. The magnetic field in the channel is determined both by the external sources and also by the currents flowing in the gas. Generally speaking, therefore, the magnetic field exterior to the channel cannot be specified arbitrarily. In the general case on the boundary of the channel we must fulfil the condition of continuity of the normal component of the magnetic field, i.e.

$$\left[\frac{\partial \Phi}{\partial s} \right] = 0 \text{ on the contour } S \tag{2.8}$$

and also the condition connecting the discontinuity in the tangential component of the magnetic field with the surface current flowing in the walls of the channel, i.e.

$$\left[\frac{\partial \Phi}{\partial n}\right] = i_x, \qquad [H_x] = i_s \text{ on the contour } S \tag{2.9}$$

Here the symbol [ ] denotes the difference between the values on the interior and exterior faces of the channel wall, i is the density of the surface current. We note that outside the channel the magnetic field satisfies the usual system of Maxwell's equations.

Besides the conditions (2.6) to (2.9), in the solution of actual problems there may arise certain supplementary conditions, establishing, for example, a connection between the constants in (2.6), or certain conditions of flow symmetry. These conditions, connected with the method of supplying current to the system and the conditions for closing the current flowing along the axis of x at infinity, need to be formulated for each actual problem (certain examples of such conditions are to be seen in (4.6) and (4.15)).

Hence, to determine the magnetic field, and consequently also the current field in the channel, we need to solve the system of Equations (2.5) with the boundary conditions (2.6) to (2.9).

3. The system (2.5) is somewhat complex. Certain particular solutions of this system are of interest. Let us indicate the properties of the simplest particular solutions of the system (2.5).

Let us consider the solutions of the system (2.5) which possess the property that

$$H_x = k\Phi, \qquad k = \text{const} \tag{3.1}$$

Under this condition the functions Φ and H_x satisfy Laplace's equation:

$$\Delta\Phi = 0, \ \Delta H_x = 0$$

From conditions (2.6) to (2.7) it follows that in this case the function Φ is subject to the following boundary conditions:

$$\Phi = \text{const} \quad \text{(on insulators)}, \qquad \frac{\partial \Phi}{\partial n} = -\alpha k \Phi \frac{\partial \Phi}{\partial s} \quad \text{(on conductors)}$$

The solutions considered correspond to the case of the absence of currents and components of electromagnetic force in the direction of the x-axis:

$$\frac{4\pi}{c} j_x = \frac{\partial H_z}{\partial y} - \frac{\partial H_y}{\partial z} = -\Delta\Phi = 0, \qquad f_x = -\frac{1}{\alpha c} j_x = 0$$

In the plane perpendicular to the axis of x, currents flow along the magnetic field

$$j_y = \frac{c}{4\pi}\frac{\partial H_x}{\partial z} = k\frac{c}{4\pi}\frac{\partial \Phi}{\partial z} = \frac{kc}{4\pi}H_y, \qquad j_z = -\frac{c}{4\pi}\frac{\partial H_x}{\partial y} = \frac{kc}{4\pi}H_z$$

We note that the solution of system (2.5) reduces to the solution of Laplace's equation for all the particular solutions satisfying the relations $H_x = f(\Phi)$, where f is an arbitrary function.

Another class of particular solutions, for which the solutions of system (2.5) satisfy simpler equations, are solutions possessing the property that

$$\triangle \Phi = kH_x, \qquad k = \mathrm{const} \tag{3.2}$$

Here to determine H_x we have the equation

$$\triangle H_x + k^2 H_x = 0 \tag{3.3}$$

Equations (3.2) and (3.3) are connected by the boundary conditions (2.6) to (2.9). In certain particular cases, for example, with $H_n|_S = 0$ the boundary conditions for Φ and H_x are formulated independently. Here it is necessary first of all to solve Equation (3.3), and then with the known function H_x to solve Equation (3.2) for the function Φ.

The solutions considered possess the property that the current density and the electromagnetic intensity in the direction of the x-axis are proportional to the intensity of the magnetic field in that direction:

$$\frac{4\pi}{c} j_x = -\triangle \Phi = -kH_x, \qquad j_x = \frac{k}{\alpha} H_x \tag{3.4}$$

Particular solutions of the types (3.1) and (3.2) can describe the solution of the problem of a discharge current with electrodes under conditions when there exists anisotropic conductivity.

Suppose, for example, that we have an infinite plane electrode occupying the plane $y = 0$. Suppose that the magnetic field outside the electrode is directed along the axis of z and that there is no surface current in the electrode. Suppose, moreover, that the normal component of current density at the electrode is a constant. Condition (2.7) and the stipulated assumptions lead to the following boundary conditions for Equations (3.2) to (3.3):

$$\frac{\partial H_x}{\partial y} = -\alpha H_x \frac{\partial H_x}{\partial z}, \qquad \frac{\partial H_x}{\partial z} = I, \qquad I = -\frac{4\pi}{c} j_y(0, z) \quad \text{when } y = 0 \tag{3.5}$$

$$\Phi = \mathrm{const} \quad \text{when } y = 0 \tag{3.6}$$

Solutions of Equation (3.3) under conditions (3.5) and Equation (3.2) with the condition (3.6) have the form

$$H_x = I\left(\cos ky - \frac{\alpha I}{k}\sin ky\right)(z + C), \qquad C = \text{const} \tag{3.7}$$
$$\Phi = \frac{I}{k}\left(\frac{\alpha I}{k}\sin ky - \cos ky\right)(z + C) + \frac{I}{k}z + H_0, \qquad H_0 = H(0, -C)$$

The constant C is related to the choice of the origin of coordinates. The solution (3.7) corresponds to the case when the external magnetic field at the electrode varies along the electrode according to the law

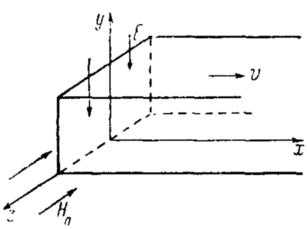

Fig. 1.

$$H_z(0, z) = -\frac{\partial\Phi}{\partial z} = -\frac{\alpha I^2}{k}(z + C) + H_0 \tag{3.8}$$

We can pose the question of the solution of the problem of the discharge current with a plane electrode in a situation similar to the foregoing, but with a homogeneous external magnetic field directed along the axis of z($H_z = H_0$ when $y = 0$). The solution of this problem will not be included amongst the solutions of types (3.1) or (3.2), but it can be constructed in the following form:

$$H_x = (1 - \alpha Iy)(Iz + C), \qquad H_z = H_0 \exp\frac{(\alpha Iy - 1)^2 - 1}{2}, \qquad H_y = 0 \tag{3.9}$$
$$\frac{4\pi}{c} j_x = H_0(\alpha Iy - 1)\alpha I \exp\frac{(\alpha Iy - 1)^2 - 1}{2}$$
$$\frac{4\pi}{c} j_y = (1 - \alpha Iy) I, \qquad \frac{4\pi}{c} j_z = \alpha I (Iz + C), \qquad C = \text{const}$$

In the solutions (3.7), (3.8) and (3.9) the current density j_y vanishes for a certain value of $y = y^*$. Accordingly, these solutions can be regarded as solutions of the problem of a discharge current with a plane electrode in a space bounded by a plane insulator. The second electrode must then be regarded as located at infinity.

4. Let us consider the problem of the effect of anisotropic conductivity on the flow of gas in a channel (Fig. 1) of rectangular shape, where two walls of the rectangle are electrodes, which are kept at a constant potential difference, and the other two walls are dielectrics. The external magnetic field is given in the yz plane ($H_x^\circ = 0$). The interaction of the currents flowing in the channel, on account of the external difference of potential, with the magnetic field, consisting of the sum of the external field and the field of the currents, leads to the creation of a force directed along the channel. Acceleration of the gas in the channel arises as a result of this force, and also on account of the Joule heating by the flow of current in the gas.

In order to determine the electromagnetic force acting on the gas, and to establish in what manner the presence of anisotropic conductivity modifies this force, let us determine the magnetic field and the field of the currents in the channel.

We shall assume that all the conditions formulated in Sections 1-2 are fulfilled. Then the solution of the problem of determining the field of the currents reduces to the solution of the system (2.5) with the boundary conditions (2.6) to (2.9), given in the appropriate form as the sides of the rectangle which represents the cross-section of the channel by the yz plane. We have not succeeded in obtaining the solution of this problem in general form, but with certain supplementary assumptions such a solution can be constructed approximately.

We shall assume that the currents flowing in the channel are relatively small, so that in the first approximation we can neglect the change in the magnetic field on account of these currents and can assume the magnetic field to be given. This assumption is equivalent to assuming that the parameter λ, introduced in (1.2), is a small quantity, i.e. the following relation holds:

$$R_m \ll \lambda < 1 \tag{4.1}$$

If the inequality (4.1) is satisfied, then the solution of system (2.5) may be sought in the form of a power series in the parameter λ (if (4.1) is not satisfied, then we can develop a method of successive approximations for the solution of the problem, and it is obvious that the first approximations in the method of successive approximations and in the series development will coincide). In what follows we shall assume that (4.1) is valid. Then it is evident that at each approximation we shall obtain from the system (2.5) equations with known coefficients for the determination of the functions H_x and $\Delta\Phi$.

For the sake of definiteness we shall assume that the external magnetic field $\mathbf{H}^\circ$ is homogeneous and parallel to the axis of $z(\mathbf{H}^\circ = H_0\mathbf{k})$. Let us introduce dimensionless variables by the formulas

$$\mathbf{H} = H_0\mathbf{H}', \qquad y = by', \qquad z = bz'$$
$$\Phi = H_0 b\Phi', \qquad \mathbf{j} = \frac{c}{4\pi}\frac{H_0}{b}\mathbf{j}', \qquad \mathbf{E} = \frac{H_0 c}{4\pi\sigma b}\mathbf{E}', \qquad \varphi = \frac{cH_0}{4\pi\sigma}\varphi'$$
$$H_0\alpha = (\omega\tau)', \qquad b = 2a, \qquad \mathbf{j} = \frac{H_0^2}{4\pi b}\mathbf{j}' \tag{4.2}$$

Here φ is the potential of the electric field ($\mathbf{E} = \mathrm{grad}\ \varphi$), b is the length of the side of the rectangle, representing the electrode, $\mathbf{j}$ is the electromagnetic force. Omitting henceforth the primes from the dimensionless quantities, we can rewrite the system (2.5) in the dimensionless variables

$$\begin{aligned} \triangle H_x + \omega\tau\left[\frac{\partial\Phi}{\partial z}\frac{\partial}{\partial y} - \frac{\partial\Phi}{\partial y}\frac{\partial}{\partial z}\right]\triangle\Phi = 0 \\ \triangle\Phi - \omega\tau\left[\frac{\partial\Phi}{\partial z}\frac{\partial}{\partial y} - \frac{\partial\Phi}{\partial y}\frac{\partial}{\partial z}\right]H_x = 0 \end{aligned} \qquad (4.3)$$

The boundary conditions (2.6) and (2.7) for the given problem in the dimensionless variables (4.2) take the form

$$H_x = \text{const} \quad \text{when } z = \pm\frac{1}{2}, \qquad 0 \leqslant y \leqslant \frac{\delta}{b} \qquad (4.4)$$

$$\frac{\partial H_x}{\partial y} = -\,\omega\tau\left[\frac{\partial\Phi}{\partial z}\triangle\Phi + H_x\frac{\partial H_x}{\partial z}\right] \quad \text{when } y = 0 \text{ and } y = \frac{\delta}{b}, \qquad -\frac{1}{2} \leqslant z \leqslant \frac{1}{2} \qquad (4.5)$$

The constants in (4.4) are determined by the total current flowing through the system. If the current supply to the system is realized according to the scheme depitcted in Fig. 2, then in place of (4.4) we have to use the relations

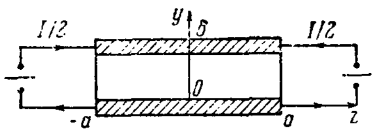

Fig. 2.

$$H_x(y, -\tfrac{1}{2}) = -H_x(y, \tfrac{1}{2})$$

$$H_x(y, -\tfrac{1}{2}) - H_x(y, \tfrac{1}{2}) = -I \qquad (4.6)$$

Here I is the total current flowing through the system ($I > 0$ if the current flows in the positive direction of the axis of y, and $I < 0$ if it flows in the opposite direction). If there are no surface currents along the axis of x in the electrodes, then conditions (2.8) and (2.9) reduce to the conditions for continuity of the magnetic field at the dielectrics and continuity of the y- and z-components of the magnetic field at the electrodes, i.e.

$$\begin{aligned} [\mathbf{H}] = 0 \quad \text{when } z = \pm\frac{1}{2}, \qquad 0 \leqslant y \leqslant \frac{\delta}{b} \\ [H_y] = [H_z] = 0 \quad \text{when } y = 0 \text{ and } y = \frac{\delta}{b}, \qquad -\tfrac{1}{2} \leqslant z \leqslant \tfrac{1}{2} \end{aligned} \qquad (4.7)$$

We shall seek a solution of the system (4.3) with the boundary conditions (4.5) to (4.6) in the form of the following series in powers of the parameter λ:

$$H_x = \lambda H_{x1} + \ldots, \quad \Phi = -y + \lambda\Phi_1 \ldots \qquad (4.8)$$

Also for determining the other quantities we obtain series, in the following forms:

$$H_z = 1 + \lambda H_{z1} + \ldots, \qquad H_y = \lambda H_{y1} + \ldots, \qquad j_x = \lambda j_{x1} + \ldots \quad \text{etc.} \tag{4.9}$$

Substituting the series (4.8) in the system (4.3) and the boundary conditions (4.5) to (4.6) and retaining only terms of the first order in λ, we obtain a system for the determination of the first approximation

$$\triangle H_{x1} + \omega\tau \frac{\partial \triangle \Phi_1}{\partial z} = 0, \qquad \triangle \Phi_1 = \omega\tau \frac{\partial H_{x1}}{\partial z}$$

$$\frac{\partial H_{x1}}{\partial y} = 0 \text{ when } y = 0, \qquad y = \delta, \qquad -\tfrac{1}{2} \leqslant z \leqslant \tfrac{1}{2}$$

$$H_{x1}(y, -\tfrac{1}{2}) = -H_{x1}(y, \tfrac{1}{2}), \; H_{x1}(y, -\tfrac{1}{2}) - H_{x1}(y, \tfrac{1}{2}) = -I_1 = \int_{-1/2}^{1/2} j_{y1}\, dz$$

It is easily seen here that the function

$$H_{x1} = I_1 z \tag{4.10}$$

satisfies this system of equations and boundary conditions. In the first approximation, then, the current density parallel to the axis of x is constant with respect to the cross-section of the channel

$$j_{x1} = -\omega\tau \frac{\partial H_x}{\partial z} = -I_1 \omega\tau \tag{4.11}$$

The current densities along the axes of y and z and the force intensity in the x direction in this approximation are respectively given by

$$j_y = \frac{\partial H_x}{\partial z} = I_1; \qquad j_z = -\frac{\partial H_x}{\partial y} = 0, \qquad f_x = I_1 \tag{4.12}$$

For the determination of the function Φ_1 in the whole of space we obtain

$$\triangle \Phi_1 = A, \qquad A = \begin{cases} \omega\tau I_1 & \text{inside the channel} \\ 0 & \text{outside the channel} \end{cases} \tag{4.13}$$

while the boundary conditions (4.7) reduce to the condition of continuity of Φ_1 at the boundary of the channel. Equation (4.13) is Poisson's equation and its solution is easily written down. We note that the distribution of the magnetic field (H_{y1}, H_{z1}) in the given case can be obtained, without solving Equation (4.13), from the Biot-Savart formula for the given distribution of current density j_{x1}. If the channel is a very elongated rectangle, then the currents in the direction of the

x-axis in the channel can be regarded as a current sheet. In these cases the expressions for Φ_1 and the component of magnetic field take the simple forms

$$\Phi_1 = \frac{\omega\tau}{2} I_1\left(y - \frac{\delta}{2b}\right)^2, \quad H_{z1} = -\omega\tau I_1\left(y - \frac{\delta}{2b}\right), \quad H_{y1} = 0 \quad \text{when } \delta \ll b \tag{4.14}$$

In obtaining the Expressions (4.14) we have made use of the condition of symmetry of the closure of the currents j_x at infinity, i.e. it is assumed that at infinity there exists a scheme for closing these currents similar to the scheme in Fig. 1 for the imposed currents. These conditions can be formulated in the following manner:

$$H_{z1}(0, z) = -H_{z1}(\delta/b, z)$$

$$H_{z1}(0, z) = -H_{z1}(\delta/b, z) = -\frac{\delta}{b} j_x = \frac{\delta}{b}\omega\tau I_1 \quad \text{when } \delta \ll b \tag{4.15}$$

Making use of (4.10), (4.13) and (4.14), we obtain from (4.3), (4.5) and (4.6) a system of equations and boundary conditions to determine the second approximation in the case $\delta \ll b$. This system is as follows:

$$\frac{\partial^2 H_{x2}}{\partial y^2} + [1 + (\omega\tau)^2]\frac{\partial^2 H_{x2}}{\partial z^2} = 0, \qquad \triangle\Phi_2 = \omega\tau\left[\frac{\partial H_{x2}}{\partial z} - \omega\tau I_1^2\left(y - \frac{\delta}{b}\right)\right]$$

$$\frac{\partial H_{x2}}{\partial y} = -\omega\tau I_1^2 z \ \text{ when } y = 0 \ \text{ and } \ y = \frac{\delta}{b}, \qquad -\tfrac{1}{2} \leqslant z \leqslant \tfrac{1}{2}$$

$$H_{x2}\left(y, -\tfrac{1}{2}\right) = -H_{x2}\left(y, \tfrac{1}{2}\right)$$

$$H_{x2}\left(y, -\tfrac{1}{2}\right) - H_{x2}\left(y, \tfrac{1}{2}\right) = -I_2 = -\int_{-1/2}^{1/2} j_{y2}\,dz \tag{4.16}$$

Accordingly, for the second approximation for the calculation of H_x we obtain a mixed problem for Laplace's equation, whilst the function Φ_2 is again determined from Poisson's equation. The solutions of the system (4.16) supply a correction to the current distribution (4.11), (4.12) with respect to the cross-section of the channel. It is obvious that taking account of the second approximation $j_z \neq 0$ and $j_y \neq$ const, i.e. allowance for the second approximation shows that the force intensity in the direction of the axis of x is not uniformly distributed across the section of the channel.

It is easy to see that the function $H_{x2}(y, z)$ satisfying (4.16) has the form

$$H_{xz}(yz) = \sum_{n=1}^{\infty} \left\{ \frac{2I_2(-1)^{n-1}}{\pi(2n-1)} \operatorname{cosech} \frac{n\pi b}{\delta\sqrt{1+(\omega\tau)^2}} \times \right.$$
$$\times \left[\sinh \frac{\pi(2n-1)b}{\delta\sqrt{1+(\omega\tau)^2}} \left(z + \frac{1}{2}\right) - \sinh \frac{\pi(2n-1)b}{\delta\sqrt{1+(\omega\tau)^2}} \left(\frac{1}{2} - z\right) \right] \times$$
$$\times \cos \frac{\pi(2n-1)b}{\delta} y + \frac{2\omega\tau I_1^2}{(2n)^2\pi^2\sqrt{1+(\omega\tau)^2}} \operatorname{cosech} \frac{2n\pi\delta\sqrt{1+(\omega\tau)^2}}{b} \times$$
$$\left. \times \left[\cosh 2n\pi\sqrt{1+(\omega\tau)^2}\, y - \cosh 2n\pi\sqrt{1+(\omega\tau)^2} \left(\frac{\delta}{b} - y\right) \right] \sin 2n\pi \left(z + \frac{1}{2}\right) \right\}$$

Leaving aside the study of the second approximation, let us consider the dependence of the total current flowing in the cross-section of the channel from the application to the electrodes of the potential difference. Making use of the projection of Equation (1.3) on the axis of y (in the dimensionless variables (4.2))

$$\frac{\partial H_x}{\partial z} = \frac{\partial \varphi}{\partial y} + \omega\tau \left[H_x \frac{\partial H_x}{\partial y} + H_z j_x \right] \tag{4.17}$$

Using the solution (4.10) to (4.12) in the first approximation, we find that

$$\varphi_1(yz) = \varphi_1(y) = \varphi(0) + [1 + (\omega\tau)^2]\, I_1 y \tag{4.18}$$

or, that the total current flowing in the cross-section of the channel is related to the potential difference applied to the electrodes by the equation

$$I_1 = \frac{\varphi(\delta/b) - \varphi(0)}{[1 + (\omega\tau)^2]\, \delta/b} \tag{4.19}$$

In dimensional variables the Equation (4.19) takes the form

$$j_{y1} = \frac{\varphi(\delta) - \varphi(0)}{\delta} \frac{\sigma}{1 + (\omega\tau)^2} \tag{4.20}$$

We note that in the expression of j_{y1} in terms of the potential difference we have used the complete difference of the potentials, i.e. in the calculation of the first approximation for the potential we have used for boundary conditions the true values of the potentials at the electrodes. Such an approach makes it possible in any approximation to obtain the connection relating the current density at the electrodes, and consequently the total current, to the applied potential difference. It is natural, in the calculation of successive approximations for the potential (φ_2, φ_3, ...), that the boundary values for these functions be taken equal to zero ($\varphi_2(0, z) = 0$, $\varphi_2(\delta/b, z) = 0$ and so on).

The Expression (4.20) shows that in the first approximation for the calculation of current density in the given problem we can make use of the usual Ohm's law, but with a modified value for the conductivity:

$$\sigma \to \frac{\sigma}{1 + (\omega\tau)^2}$$

It appears that it is possible to establish how the total current in the system for a given potential difference at the electrodes changes if we take account of the second approximation. This question can be answered without solving the system (4.16). In fact, in the second approximation Equation (4.17) gives

$$\frac{\partial H_{x2}}{\partial z} = \frac{\partial \varphi_2}{\partial y} + \omega\tau\,[j_{x2} + H_{z1} j_{x1}] \tag{4.21}$$

Substituting in this equation for j_{x1} from (4.11), for H_{z1} from (4.14) and for $j_{x2} = -\Delta\Phi_2$ from (4.16), we obtain

$$\frac{\partial \varphi_2}{\partial y} = [1 + (\omega\tau)^2]\frac{\partial H_{x2}}{\partial z} - 2\,(\omega\tau)^3 I_1{}^2\left(y - \frac{\delta}{2b}\right) \tag{4.22}$$

Integrating this equation with respect to y from zero to δ/b and making use of the boundary conditions for φ_2 ($\varphi_2(0, z) = \varphi_2(\delta/b, z) = 0$), we find that

$$\int_0^{\delta/b} \frac{\partial H_{x2}}{\partial z}\, dy = 0$$

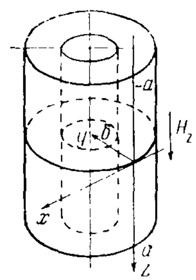

Fig. 3.

Consequently

$$\int_0^{\delta/b} I_2\, dy = I_2 \frac{\delta}{b} = \int_0^{\delta/b}\int_{-1/2}^{1/2} \frac{\partial H_{x2}}{\partial z} dy\, dz = 0 \qquad I_2 = 0 \tag{4.23}$$

Accordingly, under the specified assumptions (see (4.15)), the relation between the total current flowing in the cross-section of the channel and the applied potential difference is given to two approximations by (4.19).

We emphasize that the result obtained concerning the current density is related in essential fashion to the assumption (4.15) concerning the symmetry of the closure of currents flowing in the direction of the x-axis. Let us illustrate how the solution changes, if in place of (4.15) we take other conditions. Suppose that instead of (4.15) the following conditions hold:

$$H_{z1}(0, z) = 0, \qquad H_{z1}\left(\frac{\delta}{b}, z\right) = \frac{\delta}{b}\,\omega\tau I_1 \quad \text{when } \delta \ll b \tag{4.24}$$

Such conditions occur if we consider the problem of the flow of gas between two cylinders to which a potential difference is applied (a homopolar, Fig. 3). If the radii of the cylinders are sufficiently large and differ little enough between themselves, then the flow in the space (neglecting centrifugal effects) can be regarded as flow in a channel. For the system of two cylinders it is natural to assume that the magnetic field outside the outer cylinder is given as H_{z0}. Accordingly for the corresponding flow in the channel it is necessary to take the conditions (4.24). The flow in the channel will then not be symmetrical with respect to the plane $y = \delta/2$. Instead of (4.14), we shall for this case obtain

$$\Phi_1 = -\frac{\omega\tau}{2} I_1 y^2, \qquad H_{z1} = \omega\tau I_1 y, \qquad H_{y1} = 0 \tag{4.25}$$

Corresponding to (4.23) we obtain

$$I_2 = -\frac{(\omega\tau)^3}{1+(\omega\tau)^2} I_1^{\,2} \frac{\delta}{b} \tag{4.26}$$

Accordingly, in this case taking account of the second approximation shows that the total current flowing in the cross-section of the channel turns out to be less than the value given by Formula (4.19). It is obvious that the mean electromagnetic force acting on the gas also turns out to be less than its value calculated according to the first approximation.

5. Let us consider the problem of the flow of gas in a cylindrical channel of arbitrary shape in the plane perpendicular to the axis of x, where the external magnetic field will be assumed homogeneous and specified in the plane $yz(H_x^{\circ} = 0)$. If we limit ourselves to solution of the problem in the first approximation, then the solution reduces to the successive solution of the following equations, arising from the system (4.3):

$$\triangle H_{x1} + (\omega\tau)^2 \left[H_y^{\,2} \frac{\partial^2 H_{x1}}{\partial y^2} + 2H_y H_z \frac{\partial^2 H_{x1}}{\partial y\, \partial z} + H_z^{\,2} \frac{\partial^2 H_{x1}}{\partial z^2} \right] = 0$$

$$\triangle \Phi_1 = \omega\tau \left[H_y \frac{\partial H_{x1}}{\partial y} + H_z \frac{\partial H_{x1}}{\partial z} \right] \tag{5.1}$$

Choosing a system of coordinates so that the axis of z is directed along the magnetic field, and taking the magnitude of the magnetic field intensity as the characteristic quantity in (4.2), we reduce (5.1) to the form

$$\triangle H_{x1} + (\omega\tau)^2 \frac{\partial^2 H_{x1}}{\partial z^2} = 0, \qquad \triangle \Phi_1 = \omega\tau \frac{\partial H_{x1}}{\partial z} \tag{5.2}$$

Transformation of the coordinates according to the formulas

$$z' = \frac{z}{\sqrt{1+(\omega\tau)^2}}, \qquad y' = y \tag{5.3}$$

reduces the first Equation (5.2) to Laplace's equation

$$\triangle' H_{x1} = 0 \tag{5.4}$$

The boundary conditions (2.6) to (2.7) for the first Equation (5.2) in the variables of (4.2) have the form (H_n = cos (nz), H_s = cos (ny))

$$[1+(\omega\tau)^2 H_n{}^2]\frac{\partial H_{x1}}{\partial n} = -(\omega\tau)^2 H_n H_s \frac{\partial H_{x1}}{\partial s} \quad \text{on electrodes} \tag{5.5}$$
$$H_{x1} = \text{const} \quad \text{on insulators}$$

Accordingly, solution of the problem reduces to solution of Laplace's equation (5.4) for the boundary conditions (5.5), written in variables y', z' on the contour in which the boundary of the channel cuts the (y', z') planes.

Let us now take the projections of the equation representing Ohm's law (1.3) upon the axes of a coordinate system in which H_y = 0. In the first approximation these will have the form

$$j_{x1} = -\omega\tau\frac{\partial H_{x1}}{\partial z}, \qquad j_{y1} = \frac{\partial H_{x1}}{\partial z} = \frac{\partial \varphi_1}{\partial y} + \omega\tau j_{x1}, \qquad j_{z1} = -\frac{\partial H_{x1}}{\partial y} = \frac{\partial \varphi_1}{\partial z} \tag{5.6}$$

Hence it follows that the potential of the electric field is given in the first approximation by the equation

$$\triangle\varphi_1 = (\omega\tau^2)\frac{\partial^2 H_{x1}}{\partial y \partial z} \tag{5.7}$$

Accordingly the density of the space charge in such flows is determined to the first approximation by the relation

$$\rho_{e1} = \frac{(\omega\tau)^2}{4\pi}\frac{\partial^2 H_{x1}}{\partial y\, \partial z} \tag{5.8}$$

We note that, in the case of the rectangular channel considered in Section 4, the density of space charge is zero to the first approximation.

In order to obtain the expression for the potential satisfying the boundary conditions it is not necessary to integrate Equation (5.7), for we can make direct use of Equations (5.6). Integrating the second Equation (5.6) with respect to y, we obtain

$$\varphi_1 = [1+(\omega\tau)^2]\int \frac{\partial H_{x1}}{\partial z}\, dy + f(z)$$

To determine $f(z)$, let us substitute this expression in the third Equation (5.6) and make use of the first Equation (5.2)

$$\frac{\partial \varphi_1}{\partial z} = [1 + (\omega\tau)^2] \int \frac{\partial^2 H_{x1}}{\partial z^2} dy + f'(z) = -\int \frac{\partial^2 H_{x1}}{\partial y^2} dy + f'(z) =$$
$$= -\frac{\partial H_{x1}}{\partial y} + C_1 + f'(z) = -\frac{\partial H_{x1}}{\partial y}$$

Hence it follows that $f(z) = C_1 z + C_2$. Accordingly we have the following expression for the potential:

$$\varphi_1 = [1 + (\omega\tau)^2] \int \frac{\partial H_{x1}}{\partial z} dy + C_1 z + C_2 = -\int \frac{\partial H_{x1}}{\partial y} dz + C_3 y + C_4$$

The foregoing arguments enable us to obtain rather simply the solution in the first approximation for flow in channels of arbitrary shape.

6. For a rather more complicated example of flow in a channel than that of Section 4, let us consider the problem of flow in a channel of rectangular section, when the current is supplied to the system through "point electrodes" (Fig. 4) located at the points $z = 0$, $y = \pm\,\delta$. The external magnetic field will be assumed to be uniform and parallel to the z-axis. The problem in this case reduces to the first equation (5.2) for the function H_{x1}. If we assume that the supply of current is symmetric, then the conditions (5.5) reduce to conditions of constant H_{x1} on each of the halves of the rectangle separated by the electrodes, thus

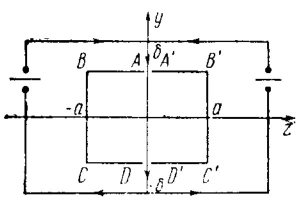

Fig. 4.

$$H_{x1}(ABCD) = -H_{x1}(A'B'C'D'), \qquad H_{x1}(ABCD) - H_{x1}(A'B'C'D') = -I_1$$

The solution of the first Equation (5.2) under these conditions has the form

$$H_{x1}(yz) = I_1 \left\{ z + \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{\cosh\,[n\pi y \sqrt{1 + (\omega\tau)^2}]}{\cosh\,[(n\pi\delta / a) \sqrt{1 + (\omega\tau)^2}]} \frac{\sin n\pi z}{n} \right\} \qquad (6.1)$$

(Here we have taken a as the characteristic linear dimension (see (4.2)).) Differentiating Equation (6.1) and making use of the formulas

$$j_{x1} = -\omega\tau \frac{\partial H_{x1}}{\partial z}, \qquad j_{y1} = \frac{\partial H_{x1}}{\partial z}, \qquad j_{z1} = -\frac{\partial H_{x1}}{\partial y}, \qquad f_{x1} = \frac{\partial H_{x1}}{\partial z}$$

we can map the distribution of currents and force intensities with respect to the channel section. The distribution of potential is given by the formula

$$\varphi_1 = [1 + (\omega\tau)^2]\, I_1 \left\{ y + \frac{2}{\pi} \frac{1}{\sqrt{1 + (\omega\tau)^2}} \sum_{n=1}^{\infty} \frac{\sinh n\pi y \sqrt{1 + (\omega\tau)^2}}{\cosh (n\pi\delta \sqrt{1 + (\omega\tau)^2} / a)} \frac{\cos n\pi z}{n} \right\} \qquad (6.2)$$

We note that in the given case there arises in the channel a space distribution of charge which is non-zero even in the calculation to the first approximation. The distribution of space charge across the section is easily obtained by making use of Formulas (5.8) and (6.1).

Translated by A.H.A.